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A state space formalism for anisotropic elasticity. Part I: Rectilinear anisotropy

Jiann-Quo Tarn *

Department of Civil Engineering, National Cheng Kung University, Tainan 70101, Taiwan, ROC

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Abstract

A state space formalism for anisotropic elasticity including the thermal effect is developed. A salient feature of the formalism is that it bridges the compliance-based and stiffness-based formalisms in a natural way. The displacement and stress components and the thermoelastic constants of a general anisotropic elastic material appear explicitly in the formulation, yet it is simple and clear. This is achieved by using the matrix notation to express the basic equations and grouping the stress in such a way that it enables us to cast neatly the three-dimensional equations of anisotropic elasticity into a compact state equation and an output equation. The homogeneous solution to the state equation for the generalized plane problem leads naturally to the eigen relation and the sextic equation of Stroh. Extension, twisting, bending, temperature change and body forces are accounted for through the particular solution. Based on the formalism the general solution for generalized plane strain and generalized torsion of an anisotropic elastic body are determined in an elegant manner.

Keywords: Anisotropic elasticity; Rectilinear anisotropy; Generalized plane strain; Generalized torsion; Temperature change; State space formalism

1. Introduction

Plane problems of anisotropic elasticity were studied by Lekhnitskii (1968, 1981) using a compliance-based formalism, and by Eshelby et al. (1953), Stroh (1958), Ting (1996, 2000) among others, using a stiffness-based formalism. According to the Lekhnitskii formalism, the stress is expressed in terms of a pair of stress functions such that the equilibrium equations are satisfied identically, and through the compatibility conditions a system of high order differential equations for the stress functions is derived to determine the solution. In the Stroh formalism the general solution is expressed in terms of the eigenvectors and analytic functions of complex variables, and the matrix identities derived from the eigen relations are useful in simplifying or interpreting the results. It is believed that the two formalisms are equivalent, yet this was

* Fax: +886-6-2358542.
E-mail address: jqtarn@mail.ncku.edu.tw (J.-Q. Tarn).

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taken for granted until Barnett and Kirchner (1997) gave a formal proof of the equivalence of the sextic equations in the two formalisms. Recently, Yin (2000a,b) studied the latent structure of the Lekhnitskii formalism and showed their duality by making use of the eigen relations. In both formalisms the stress functions play an essential role. It should be noted that the Stroh formalism does not allow for antiplane deformations associated with extension, torsion and bending.

In this paper we develop a state space formalism for anisotropic elasticity including the thermal effect. The paper consists of two parts. Part I presents the formalism for the rectilinearly anisotropic body in which the elastic property at each point is characterized by the three directions in the Cartesian coordinates. Beginning with a three-dimensional formulation, we show that the compliance-based and stiffness-based formulations result in the same state equation and output equation, and the homogeneous solution to the state equation for the generalized plane problem leads naturally to the eigen relation and the sextic equation of Stroh. The two formalisms coincide in the state space framework.

The idea of the state space has been used extensively in the system engineering and control theory. In the area of elasticity, Bahar (1975) showed that the plane stress problem of an isotropic elastic body could be brought within the state space framework. The basic structure was exhibited but no further development were given. Zhong (1995) presented the plane elasticity in state space and examined its connections to the Hamiltonian system. Recently, we have employed the state space approach for various problems of anisotropic materials (Wang et al., 2000; Tarn, 2001; Tarn and Wang, 2001). Herein we develop the formalism for a general anisotropic elastic body with emphasis on generalized plane strain and generalized torsion. When a general anisotropic elastic body is subjected to loadings that do not vary along the x_3 axis, the stress and strain are independent of x_3, but the displacement depends on x_3 as well as on x_1 and x_2. The body is in the state of *generalized plane strain* when subjected to an axial force and bending moments at the ends and surface tractions that are independent of x_3; it is in the state of *generalized torsion* when subjected to a torque at the ends and free from surface tractions and body forces (Lekhnitskii, 1981). The two classes of problems are referred to as the *generalized plane problem*. They differ only in the boundary conditions, and can be treated in the same context.

The underlying concept of the state space formalism is that the elastic body is regarded as a linear system. The state equation is derived from the basic equations without eliminating the stress or the displacement so that it involves at most second derivatives of the field variables. To be concise, it is expedient to group the stress into two parts: $\boldsymbol{\tau}_2 = [\sigma_{12}, \sigma_{23}, \sigma_{22}]$, consisting of the components with one of the subscripts being 2, and the remaining components, $\boldsymbol{\tau}_1 = [\sigma_{11}, \sigma_{33}, \sigma_{13}]$. The reason for making this grouping is that for the problems in Cartesian coordinates, such as stress analysis of a laminated system or a layered medium, if the x_2 axis is pointed in the thickness direction, the traction boundary conditions and the interfacial continuity conditions are directly associated with $\boldsymbol{\tau}_2$, the traction vector on the plane perpendicular to the x_2 axis. The grouping enables us to use the matrix notation to cast the three-dimensional equations of anisotropic elasticity into a compact state equation and an output equation. Related works by Alshits and Kirchner (1995a,b) on multilayers also studied the problem by representing the elasticity equations by a system of first-order differential equations. What distinguishes the present formalism from the others is that the displacement and stress and the thermoelastic constants of the anisotropic material appear explicitly without using the stress functions or the reduced material parameters (such as the reduced elastic compliances in the Lekhnitskii formalism), yet the formulation is remarkably simple. Only the displacement vector $\mathbf{u}$, $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and four matrices that characterize the elastic property of the material come into play, instead of the individual components of the displacement and stress and the twenty-one elastic constants of a general anisotropic material. Moreover, $\mathbf{u}$ and $\boldsymbol{\tau}_2$ are the only unknowns in the state equation. When applied to the generalized plane problem, it enables us to determine the general solution systematically. The coincidence of the compliance-based and stiffness-based formalisms is established by using the identities inherent in the constitutive matrices. Extension, twisting, bending, temperature change and body forces are clearly expressed by the non-homogeneous terms in the state equation. The homogeneous solution of the state equation takes the form of analytic functions of complex variables. The solution process leads, in a natural and logical manner, to the eigen relation and the sextic equation of Stroh. The particular solution takes into account the effects of the antiplane and bending deformations as well as the end loads and temperature change. In finding the solution for a specific problem the solution approaches documented in Lekhnitskii (1981) and Ting (1996) can be employed as well.

This part of the paper focuses on the development of the state space formalism in the Cartesian coordinates. Illustrative examples will be given in Part II of the paper (Tarn, 2002a) and an accompanied paper on piezothermoelasticity (Tarn, 2002b).

2. State space formulation

2.1. Basic equations in matrix form

The thermoelastic constitutive equations of an anisotropic material are

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36}\\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46}\\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56}\\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix} - \begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\\ \beta_6\end{bmatrix} T, \tag{1}$$

where σ_{ij} and ε_{ij} are the stress and strain tensors; c_{ij} and β_i are the elastic constants and the thermal coefficients of the material, respectively; T is the temperature change.

The formulation of anisotropic elasticity could be greatly simplified if the stress and strain are grouped properly. To this end, let us group the stress and strain components into two parts: one consists of the components associated with the subscript 2, another consists of the remaining components, and rewrite Eq. (1) concisely as

$$\begin{bmatrix}\boldsymbol{\tau}_1\\ \boldsymbol{\tau}_2\end{bmatrix} = \begin{bmatrix}\mathbf{C}_{11} & \mathbf{C}_{12}\\ \mathbf{C}_{12}^T & \mathbf{C}_{22}\end{bmatrix} \begin{bmatrix}\boldsymbol{\gamma}_1\\ \boldsymbol{\gamma}_2\end{bmatrix} - \begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{bmatrix} T, \tag{2}$$

where

$$\boldsymbol{\tau}_1 = [\sigma_{13} \quad \sigma_{11} \quad \sigma_{33}]^T, \qquad \boldsymbol{\tau}_2 = [\sigma_{12} \quad \sigma_{22} \quad \sigma_{23}]^T,$$
$$\boldsymbol{\gamma}_1 = [2\varepsilon_{13} \quad \varepsilon_{11} \quad \varepsilon_{33}]^T, \qquad \boldsymbol{\gamma}_2 = [2\varepsilon_{12} \quad \varepsilon_{22} \quad 2\varepsilon_{23}]^T.$$

$$\mathbf{C}_{11} = \mathbf{C}_{11}^T = \begin{bmatrix} c_{55} & c_{15} & c_{35}\\ c_{15} & c_{11} & c_{13}\\ c_{35} & c_{13} & c_{33}\end{bmatrix}, \qquad \mathbf{C}_{22} = \mathbf{C}_{22}^T = \begin{bmatrix} c_{66} & c_{26} & c_{46}\\ c_{26} & c_{22} & c_{24}\\ c_{46} & c_{24} & c_{44}\end{bmatrix},$$

$$\mathbf{C}_{12} = \begin{bmatrix} c_{56} & c_{25} & c_{45}\\ c_{16} & c_{12} & c_{14}\\ c_{36} & c_{23} & c_{34}\end{bmatrix}, \qquad \boldsymbol{\beta}_1 = \begin{bmatrix}\beta_5\\ \beta_1\\ \beta_3\end{bmatrix}, \qquad \boldsymbol{\beta}_2 = \begin{bmatrix}\beta_6\\ \beta_2\\ \beta_4\end{bmatrix}.$$

The strain–displacement relations may be expressed as

$$\begin{bmatrix}\boldsymbol{\gamma}_1\\ \boldsymbol{\gamma}_2\end{bmatrix} = \begin{bmatrix}\mathbf{L}_1\mathbf{u}\\ \mathbf{L}_2\mathbf{u}\end{bmatrix} + \partial_2 \begin{bmatrix}\mathbf{0}\\ \mathbf{u}\end{bmatrix}, \tag{3}$$

where ∂_i stands for the partial derivative with respect to x_i, and

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3]^T, \qquad \mathbf{L}_1 = \mathbf{K}_1\partial_1 + \mathbf{K}_2\partial_3, \qquad \mathbf{L}_2 = \mathbf{K}_3\partial_1 + \mathbf{K}_4\partial_3,$$

$$\mathbf{K}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{K}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{K}_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{K}_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The equilibrium equations can be written in a single matrix equation as

$$\partial_2\boldsymbol{\tau}_2 + \mathbf{L}_1^T\boldsymbol{\tau}_1 + \mathbf{L}_2^T\boldsymbol{\tau}_2 + \mathbf{F} = \mathbf{0}, \tag{4}$$

where $\mathbf{F} = [F_1 \quad F_2 \quad F_3]^T$ is the body force vector.

With the basic equations so expressed, the individual elastic constants and the displacement and stress components are no longer in view; they are replaced by $\mathbf{C}_{ij}$, $\mathbf{u}$, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ which play the principal roles hereafter.

2.2. Stiffness-based formulation

The formalism may be developed by expressing the strain by the stress through the constitutive equations in terms of either the elastic stiffness or the elastic compliance. We begin with the formulation using the stiffness representation. A key step of the state space formalism is to express the basic equations into a state equation in terms of the state vector. For the reasons aforementioned, we choose $[\mathbf{u}, \boldsymbol{\tau}_2]^T$ to be the state vector.

Substitution of Eq. (3) into Eq. (2) leads to

$$\begin{bmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{bmatrix} = \begin{bmatrix} (\mathbf{C}_{11}\mathbf{L}_1 + \mathbf{C}_{12}\mathbf{L}_2)\mathbf{u} \\ (\mathbf{C}_{12}^T\mathbf{L}_1 + \mathbf{C}_{22}\mathbf{L}_2)\mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{12}\partial_2\mathbf{u} \\ \mathbf{C}_{22}\partial_2\mathbf{u} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} T. \tag{5}$$

Eq. $(5)_2$ may be rewritten as

$$\partial_2\mathbf{u} = \begin{bmatrix} -\mathbf{C}_{22}^{-1}\mathbf{C}_{12}^T\mathbf{L}_1 - \mathbf{L}_2 & \mathbf{C}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\tau}_2 \end{bmatrix} + \mathbf{C}_{22}^{-1}\boldsymbol{\beta}_2 T. \tag{6}$$

Substituting Eq. (6) in Eq. $(5)_1$ gives the output equation:

$$\boldsymbol{\tau}_1 = \begin{bmatrix} \widetilde{\mathbf{C}}_{11}\mathbf{L}_1 & \mathbf{C}_{12}\mathbf{C}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\tau}_2 \end{bmatrix} - \tilde{\boldsymbol{\beta}}_1 T, \tag{7}$$

where

$$\widetilde{\mathbf{C}}_{11} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{12}^T, \qquad \tilde{\boldsymbol{\beta}}_1 = \boldsymbol{\beta}_1 - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\boldsymbol{\beta}_2.$$

Inserting Eq. (7) into Eq. (4) yields

$$\partial_2\boldsymbol{\tau}_2 = \begin{bmatrix} -\mathbf{L}_1^T\widetilde{\mathbf{C}}_{11}\mathbf{L}_1 & \mathbf{D}_{11}^T \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\tau}_2 \end{bmatrix} + \mathbf{L}_1^T\tilde{\boldsymbol{\beta}}_1 T - \mathbf{F}. \tag{8}$$

Eqs. (6) and (8) can be cast into a single matrix differential equation as

$$\frac{\partial}{\partial x_2}\begin{bmatrix} \mathbf{u} \\ \boldsymbol{\tau}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{C}_{22}^{-1} \\ \mathbf{D}_{21} & \mathbf{D}_{11}^T \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\tau}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{22}^{-1}\boldsymbol{\beta}_2 \\ \mathbf{L}_1^T\tilde{\boldsymbol{\beta}}_1 \end{bmatrix} T - \begin{bmatrix} \mathbf{0} \\ \mathbf{F} \end{bmatrix}, \tag{9}$$

where

$$\mathbf{D}_{11} = -\mathbf{C}_{22}^{-1}\mathbf{C}_{12}^T\mathbf{L}_1 - \mathbf{L}_2, \qquad \mathbf{D}_{21} = -\mathbf{L}_1^T\widetilde{\mathbf{C}}_{11}\mathbf{L}_1.$$

Eq. (9) is the state equation of three-dimensional anisotropic elasticity. It is central to the formalism; once it is solved, all the displacement and stress components follow.

2.3. Compliance-based formulation

The constitutive equation may also be expressed in terms of the elastic compliance as

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{12}^T & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{bmatrix} T, \tag{10}$$

where the stiffness matrix and the thermal expansion coefficients are

$$\begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{12}^T & \mathbf{S}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{12}^T & \mathbf{C}_{22} \end{bmatrix}^{-1}, \qquad \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{12}^T & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix}.$$

Substituting Eq. (3) into Eq. (10) gives

$$\begin{bmatrix} \mathbf{D}_1\mathbf{u} \\ \mathbf{D}_2\mathbf{u} \end{bmatrix} + \partial_2 \begin{bmatrix} \mathbf{0} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{12}^T & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{bmatrix} T. \tag{11}$$

Eqs. (4) and (11) can be expressed in terms of $[\mathbf{u}, \boldsymbol{\tau}_2]^T$. The output equation is obtained from Eq. $(11)_1$ as

$$\boldsymbol{\tau}_1 = \begin{bmatrix} \mathbf{S}_{11}^{-1}\mathbf{L}_1 & -\mathbf{S}_{11}^{-1}\mathbf{S}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\tau}_2 \end{bmatrix} - \mathbf{S}_{11}^{-1}\boldsymbol{\alpha}_1 T. \tag{12}$$

Substituting Eq. (12) in Eqs. $(11)_2$ and (4) yields the state equation in terms of the elastic compliance:

$$\frac{\partial}{\partial x_2} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\tau}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{D}}_{11} & \widetilde{\mathbf{D}}_{12} \\ \widetilde{\mathbf{D}}_{21} & \widetilde{\mathbf{D}}_{11}^T \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\tau}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\alpha}_2 - \mathbf{S}_{12}^T\mathbf{S}_{11}^{-1}\boldsymbol{\alpha}_1 \\ \mathbf{L}_1^T\mathbf{S}_{11}^{-1}\boldsymbol{\alpha}_1 \end{bmatrix} T - \begin{bmatrix} \mathbf{0} \\ \mathbf{F} \end{bmatrix}, \tag{13}$$

where

$$\begin{bmatrix} \widetilde{\mathbf{D}}_{11} & \widetilde{\mathbf{D}}_{12} \\ \widetilde{\mathbf{D}}_{21} & \widetilde{\mathbf{D}}_{11}^T \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{12}^T\mathbf{S}_{11}^{-1}\mathbf{L}_1 - \mathbf{L}_2 & \mathbf{S}_{22} - \mathbf{S}_{12}^T\mathbf{S}_{11}^{-1}\mathbf{S}_{12} \\ -\mathbf{L}_1^T\mathbf{S}_{11}^{-1}\mathbf{L}_1 & \mathbf{L}_1^T\mathbf{S}_{11}^{-1}\mathbf{S}_{12} - \mathbf{L}_2^T \end{bmatrix}.$$

3. Matrix identities

In order to show that the stiffness-based and compliance-based formalisms are completely equivalent, it is necessary to establish certain identities associated with the constitutive matrices. Since the compliance and stiffness matrices are inverse to each other, there exist the following relations between the sub-matrices:

$$\mathbf{S}_{11}\mathbf{C}_{11} + \mathbf{S}_{12}\mathbf{C}_{12}^T = \mathbf{I}, \qquad \mathbf{S}_{11}\mathbf{C}_{12} + \mathbf{S}_{12}\mathbf{C}_{22} = \mathbf{0}, \tag{14}$$

$$\mathbf{S}_{12}^T\mathbf{C}_{11} + \mathbf{S}_{22}\mathbf{C}_{12}^T = \mathbf{0}, \qquad \mathbf{S}_{12}^T\mathbf{C}_{12} + \mathbf{S}_{22}\mathbf{C}_{22} = \mathbf{I}. \tag{15}$$

Substituting Eq. $(14)_2$ in Eq. $(15)_2$ leads to

$$(\mathbf{S}_{22} - \mathbf{S}_{12}^T\mathbf{S}_{11}^{-1}\mathbf{S}_{12})\mathbf{C}_{22} = \mathbf{I}. \tag{16}$$

Thus

$$\mathbf{S}_{22} - \mathbf{S}_{12}^T\mathbf{S}_{11}^{-1}\mathbf{S}_{12} = \mathbf{C}_{22}^{-1}. \tag{17}$$

From Eq. (14)

$$\mathbf{S}_{11}^{-1} = \mathbf{C}_{11} + \mathbf{S}_{11}^{-1}\mathbf{S}_{12}\mathbf{C}_{12}^{T}, \qquad \mathbf{S}_{11}^{-1}\mathbf{S}_{12} = -\mathbf{C}_{12}\mathbf{C}_{22}^{-1}. \tag{18}$$

Substituting Eq. $(18)_2$ in Eq. $(18)_1$, and taking transpose of Eq. $(18)_2$ produces

$$\mathbf{S}_{11}^{-1} = \widetilde{\mathbf{C}}_{11} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{12}^{T}, \tag{19}$$

$$\mathbf{S}_{12}^{T}\mathbf{S}_{11}^{-1} = -\mathbf{C}_{22}^{-1}\mathbf{C}_{12}^{T}. \tag{20}$$

It follows from Eqs. (17), (19) and (20) that

$$\boldsymbol{\alpha}_2 - \mathbf{S}_{12}^{T}\mathbf{S}_{11}^{-1}\boldsymbol{\alpha}_1 = \mathbf{C}_{22}^{-1}\boldsymbol{\beta}_2, \qquad \mathbf{S}_{11}^{-1}\boldsymbol{\alpha}_1 = \tilde{\boldsymbol{\beta}}_1 = \boldsymbol{\beta}_1 - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\boldsymbol{\beta}_2. \tag{21}$$

These identities show that

$$\begin{bmatrix} \widetilde{\mathbf{D}}_{11} & \widetilde{\mathbf{D}}_{12} \\ \widetilde{\mathbf{D}}_{21} & \widetilde{\mathbf{D}}_{11}^{T} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{C}_{22}^{-1} \\ \mathbf{D}_{21} & \mathbf{D}_{11}^{T} \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{\alpha}_2 - \mathbf{S}_{12}^{T}\mathbf{S}_{11}^{-1}\boldsymbol{\alpha}_1 \\ \mathbf{L}_1^{T}\mathbf{S}_{11}^{-1}\boldsymbol{\alpha}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{22}^{-1}\boldsymbol{\beta}_2 \\ \mathbf{L}_1^{T}\tilde{\boldsymbol{\beta}}_1 \end{bmatrix}.$$

The coincidence of Eqs. (12), (13) and Eqs. (7), (9) thus is proved. If not for the concise matrix representation of the constitutive equations, it would be difficult to show that the stiffness-based and compliance-based formalisms are identical.

4. Generalized plane problems

4.1. State equation

As is well known, it is extremely difficult to determine the general solution to the three-dimensional equations of anisotropic elasticity, yet the structure of Eq. (9) suggests that if the functional dependence on one of the coordinates is known a priori, the equation becomes two-dimensional and an analytic solution may be obtainable. Indeed, for the generalized plane problems the formalism allows for a general solution.

When an anisotropic elastic body is in the state of generalized plane strain and generalized torsion, the stress and strain are independent of x_3. The displacement field is dependent on x_3, given by

$$u_1 = u - b_1x_3^2/2 - \vartheta x_2x_3 - \omega_3x_2 + \omega_2x_3 + u_0, \tag{22}$$

$$u_2 = v - b_2x_3^2/2 + \vartheta x_1x_3 + \omega_3x_1 - \omega_1x_3 + v_0, \tag{23}$$

$$u_3 = w + (b_1x_1 + b_2x_2 + \varepsilon)x_3 - \omega_2x_1 + \omega_1x_2 + w_0, \tag{24}$$

where u, v, w are unknown functions of x_1 and x_2; ω_1, ω_2, ω_3 and u_0, v_0, w_0 are constants characterizing the rigid body displacements. The constant ε is a uniform extension, ϑ is associated with the curvature due to twisting, b_1 and b_2 are associated with the curvatures due to bending. Eqs. (22)–(24) have been derived in Section 18 of Lekhnitskii's monograph (1981) by a rather tedious manipulation of the displacement–stress relations. The derivation based on the present formalism is simpler. It is given in Appendix A.

On substituting Eqs. (22)–(24) in Eqs. (7) and (9), the state equation and the output equation for the generalized plane problem read

$$\frac{\partial}{\partial x_2}\begin{bmatrix} \tilde{\mathbf{u}} \\ \boldsymbol{\tau}_2 \end{bmatrix} = \begin{bmatrix} -\mathbf{C}_{22}^{-1}\mathbf{A}_1\partial_1 & \mathbf{C}_{22}^{-1} \\ -\mathbf{A}_2\partial_{11} & -\mathbf{A}_1^{T}\mathbf{C}_{22}^{-1}\partial_1 \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{u}} \\ \boldsymbol{\tau}_2 \end{bmatrix} - \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{22}^{-1}\boldsymbol{\beta}_2 \\ \mathbf{L}_1^{T}\tilde{\boldsymbol{\beta}}_1 \end{bmatrix}T - \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}, \tag{25}$$

$$\boldsymbol{\tau}_1 = \begin{bmatrix} \widetilde{\mathbf{C}}_{11}\mathbf{K}_1\partial_1 & \mathbf{C}_{12}\mathbf{C}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{u}} \\ \boldsymbol{\tau}_2 \end{bmatrix} + \widetilde{\mathbf{C}}_{11}[(\varepsilon + b_1x_1 + b_2x_2)\mathbf{k}_1 - \vartheta x_2\mathbf{k}_2] - \tilde{\boldsymbol{\beta}}_1 T, \tag{26}$$

where $T = T(x_1, x_2)$, the body force F_3 is not present, and

$$\tilde{\mathbf{u}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix}, \qquad \mathbf{k}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{k}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \widetilde{\mathbf{C}}_{11} = \begin{bmatrix} \tilde{c}_{55} & \tilde{c}_{15} & \tilde{c}_{35} \\ \tilde{c}_{15} & \tilde{c}_{11} & \tilde{c}_{13} \\ \tilde{c}_{35} & \tilde{c}_{13} & \tilde{c}_{33} \end{bmatrix},$$

$$\mathbf{A}_1 = \mathbf{C}_{12}^T\mathbf{K}_1 + \mathbf{C}_{22}\mathbf{K}_3, \qquad \mathbf{A}_2 = \mathbf{K}_1^T\widetilde{\mathbf{C}}_{11}\mathbf{K}_1,$$

$$\mathbf{p}_1 = \mathbf{C}_{22}^{-1}\mathbf{C}_{12}^T[(\varepsilon + b_1x_1 + b_2x_2)\mathbf{k}_1 - \vartheta x_2\mathbf{k}_2] + \vartheta x_1\mathbf{k}_1, \qquad \mathbf{p}_2 = b_1\begin{bmatrix} \tilde{c}_{13} & 0 & \tilde{c}_{35} \end{bmatrix}^T.$$

When the stress is independent of x_3, the stress resultants over the cross section Ω reduce to an axial force P_z, a torque M_t, and bi-axial bending moments M_1, M_2; the resultant shears vanish identically. The end conditions may be written in the matrix form as

$$\int_\Omega (\mathbf{H}_1\boldsymbol{\tau}_1 + \mathbf{H}_2\boldsymbol{\tau}_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \mathbf{P}, \tag{27}$$

where

$$\mathbf{H}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & x_2 \\ 0 & 0 & -x_1 \\ -x_2 & 0 & 0 \end{bmatrix}, \qquad \mathbf{H}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x_1 \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} P_z \\ M_1 \\ M_2 \\ M_t \end{bmatrix}.$$

The non-homogeneous term in the state equation contains the constants b_1, b_2, ε and ϑ. As there is a one to one correspondence between them and the applied loads through the end conditions, they may be regarded as known a priori in the formulation.

4.2. Homogeneous solution

The general solution to Eq. (25) consists of the homogeneous solution and the particular solution. Let us consider the homogeneous solution in the form

$$\tilde{\mathbf{u}} = \mathbf{U}F(z), \qquad \boldsymbol{\tau}_2 = \mathbf{S}F'(z), \tag{28}$$

where $\mathbf{U}$ and $\mathbf{S}$ are unknown constant vectors, p is a parameter to be determined, and

$$F'(z) = \mathrm{d}F(z)/\mathrm{d}z, \quad z = x_1 + px_2.$$

Substituting Eq. (28) in Eq. (25) yields the eigen relation:

$$\begin{bmatrix} -\mathbf{C}_{22}^{-1}\mathbf{A}_1 & \mathbf{C}_{22}^{-1} \\ -\mathbf{A}_2 & -\mathbf{A}_1^T\mathbf{C}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{S} \end{bmatrix} = p\begin{bmatrix} \mathbf{U} \\ \mathbf{S} \end{bmatrix}, \tag{29}$$

where p is the eigenvalue, $[\mathbf{U}, \mathbf{S}]^T$ is the eigenvector.

Expressing $\mathbf{S}$ in terms of $\mathbf{U}$ using Eq. $(29)_1$ gives

$$\mathbf{S} = (\mathbf{A}_1 + p\mathbf{C}_{22})\mathbf{U}. \tag{30}$$

Substituting Eq. (30) in Eq. $(29)_2$ leads to

$$[\mathbf{A}_3 + p(\mathbf{A}_1 + \mathbf{A}_1^T) + p^2\mathbf{C}_{22}]\mathbf{U} = \mathbf{0}, \tag{31}$$

where

$$\mathbf{A}_3 = \mathbf{K}_1^T\mathbf{C}_{11}\mathbf{K}_1 + \mathbf{K}_1^T\mathbf{C}_{12}\mathbf{K}_3 + \mathbf{K}_3^T\mathbf{C}_{12}^T\mathbf{K}_1 + \mathbf{K}_3^T\mathbf{C}_{22}\mathbf{K}_3.$$

For the existence of a non-trivial solution of Eq. (31) the determinant of the coefficient matrix must vanish,

$$|\mathbf{A}_3 + p(\mathbf{A}_1 + \mathbf{A}_1^T) + p^2\mathbf{C}_{22}| = 0. \tag{32}$$

A simple manipulation shows that

$$\mathbf{A}_1 = \begin{bmatrix} c_{16} & c_{66} & c_{56} \\ c_{12} & c_{26} & c_{25} \\ c_{14} & c_{46} & c_{45} \end{bmatrix}, \qquad \mathbf{A}_3 = \begin{bmatrix} c_{11} & c_{16} & c_{15} \\ c_{16} & c_{66} & c_{56} \\ c_{15} & c_{56} & c_{55} \end{bmatrix},$$

so that $\mathbf{A}_3 = \mathbf{Q}$, $\mathbf{A}_1^T = \mathbf{R}$, $\mathbf{C}_{22} = \mathbf{T}$, where $\mathbf{Q}, \mathbf{R}, \mathbf{T}$ are the notations used in the Stroh formalism (Ting, 1996). This shows that Eq. (32) is precisely the sextic equation of Stroh. Moreover, the eigen relation, Eq. (29), is exactly the one posed in the Stroh formalism. Here it arises naturally and logically. The entities of the matrix are expressed in terms of the sub-matrices of the elastic stiffness.

At this stage, it is evident that the development may be carried on following the same line as in Ting's monograph. It is known that the p cannot be real by virtue of the positive-definiteness of the strain energy, and there are three pairs of complex conjugate p since the coefficients of Eq. (32) are real.

Denoting the eigenvalues and the associated eigenvectors by

$$p_k = \varkappa_k + \mathrm{i}\eta_k, \quad p_{k+3} = \bar{p}_k = \varkappa_k - \mathrm{i}\eta_k \qquad (b_k > 0), \tag{33}$$

$$\mathbf{U}_{k+3} = \overline{\mathbf{U}}_k, \quad \mathbf{S}_{k+3} = \overline{\mathbf{S}}_k \qquad (k = 1, 2, 3), \tag{34}$$

where i is the imaginary number, $\varkappa_k$ and η_k are real, there follow

$$\tilde{\mathbf{u}} = 2\mathrm{Re}\left\{ \sum_{k=1}^{3} \mathbf{U}_k F_k(z_k) \right\}, \tag{35}$$

$$\boldsymbol{\tau}_1 = 2\mathrm{Re}\left\{ \sum_{k=1}^{3} (\mathbf{A}_4 + p_k\mathbf{C}_{12})\mathbf{U}_k F_k'(z_k) \right\}, \tag{36}$$

$$\boldsymbol{\tau}_2 = 2\mathrm{Re}\left\{ \sum_{k=1}^{3} (\mathbf{A}_1 + p_k\mathbf{C}_{22})\mathbf{U}_k F_k'(z_k) \right\}, \tag{37}$$

where the $\mathbf{U}_k$ are the eigenvectors associated with p_k; F_k are functions of $z_k(= x_1 + p_k x_2)$, and

$$\mathbf{A}_4 = \mathbf{C}_{11}\mathbf{K}_1 + \mathbf{C}_{12}\mathbf{K}_3 = \begin{bmatrix} c_{15} & c_{56} & c_{55} \\ c_{11} & c_{16} & c_{15} \\ c_{13} & c_{36} & c_{35} \end{bmatrix}.$$

We remark that we have considered only mathematically non-degenerate materials such that the roots of the sextic equation are distinct. For the case of repeated roots the homogeneous solution to Eq. (25) must be modified accordingly.

4.3. Particular solution

The non-homogeneous terms in Eq. (25) are due to extension, torsion, bending, temperature change and body forces. The particular solution corresponding to a prescribed set of b_1, b_2, ε, ϑ and a given temperature change and body force can be determined in an elementary way. For a uniform temperature change ΔT and a constant body force $\mathbf{f}$ the particular solution is found to be

$$\tilde{\mathbf{u}} = \mathbf{a}_1 x_1^2/2 + \mathbf{a}_2 x_1 x_2 + \mathbf{a}_3 x_2^2/2, \tag{38}$$

$$\boldsymbol{\tau}_2 = \varepsilon \mathbf{C}_{12}^T \mathbf{k}_1 - \boldsymbol{\beta}_2 \Delta T, \tag{39}$$

$$\boldsymbol{\tau}_1 = \varepsilon \mathbf{C}_{11} \mathbf{k}_1 - \vartheta \widetilde{\mathbf{C}}_{11} \mathbf{k}_2 x_2 + \widetilde{\mathbf{C}}_{11} [(\mathbf{K}_1 \mathbf{a}_1 + b_1 \mathbf{k}_1) x_1 + (\mathbf{K}_1 \mathbf{a}_2 + b_2 \mathbf{k}_1) x_2] - \boldsymbol{\beta}_1 \Delta T, \tag{40}$$

in which the constant vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are determined from

$$\mathbf{K}_1^T \widetilde{\mathbf{C}}_{11} \mathbf{K}_1 \mathbf{a}_1 = -\mathbf{p}_2 - \mathbf{f}, \tag{41}$$

$$\mathbf{A}_1 \mathbf{a}_1 + \mathbf{C}_{22} \mathbf{a}_2 = -b_1 \mathbf{C}_{12}^T \mathbf{k}_1 - \vartheta \mathbf{C}_{22} \mathbf{k}_1, \tag{42}$$

$$\mathbf{A}_1 \mathbf{a}_2 + \mathbf{C}_{22} \mathbf{a}_3 = -b_2 \mathbf{C}_{12}^T \mathbf{k}_1 + \vartheta \mathbf{C}_{12}^T \mathbf{k}_2. \tag{43}$$

It is known that the Stroh formalism is intended for plane deformations—problems with antiplane as well as inplane deformations (for example, torsion and bending) can only be dealt with by means of superposition of various special solutions. In the state space formalism, by contrast, the effects of extension, torsion, bending, temperature change and body forces are taken into account through the particular solution systematically.

The general solution is obtained by superposing Eqs. (35)–(37) and (38)–(40) along with Eqs. (22)–(24), in which the analytic functions $F_k(z_k)$ are to be determined for a specific problem. For a simply or doubly connected domain it is often effective by using the Cauchy integral formula to derive the analytic functions from the boundary conditions, or assuming a power series representation

$$F_k(z_k) = A_k \ln z_k + \sum_{n=-\infty}^{\infty} c_{nk} z_k^n, \qquad F_k'(z_k) = A_k z_k^{-1} + \sum_{n=-\infty}^{\infty} n c_{nk} z_k^{n-1} \tag{44}$$

combined with conformal mapping to find a series solution. The solution approach has been documented in Lekhnitskii (1981) and Ting (1996).

5. Closing remarks

In the formulation the stresses are grouped into $\boldsymbol{\tau}_1 = [\sigma_{13}, \sigma_{11}, \sigma_{33}]$ and $\boldsymbol{\tau}_2 = [\sigma_{12}, \sigma_{22}, \sigma_{23}]$, the derivative with respect to x_2 is taken to the left-hand side of the state equation. Naturally, it is permissible to group the stress differently and take the derivative with respect to x_1 or x_3 to the left-hand side. In Appendix B we present an alternative formulation based on grouping the stress into inplane and antiplane components. The derivation is essentially the same as given in the main body of the text. In determining the homogeneous solution to the state equation for the generalized plane problem, the formulation again leads to the sextic equation of Stroh but the eigen relation does not emerge. We remark that any proper grouping is admissible, resulting in a state equation and an output equation different in form but same in effect.

For a laminated plate or multilayered medium the interfacial and boundary conditions along the planes $x_2 =$ constant are directly related to the state vector $[\mathbf{u}, \boldsymbol{\tau}_2]$. The formalism in conjunction with the method of transfer matrix is useful in satisfying these conditions. The transfer matrix transmits the state vector from one layer to another and takes into account the interfacial continuity and lateral boundary conditions in a systematic way. Relevant studies in the cylindrical coordinates have been presented (Tarn, 2001; Tarn and Wang, 2001). Problems in the Cartesian coordinates can be studied along such lines.

Strictly speaking, the state of generalized plane strain and generalized torsion exist only in an infinitely long prismatic body subjected to loadings that do not vary axially. Stress disturbance inevitably occurs if the exact end conditions are approximated by the statically equivalent conditions of the stress resultants. It is known that the Saint-Venant end effects in anisotropic elastic materials may not be local to the boundary layer region (Horgan, 1996). Study of the stress decay in anisotropic laminates in the state space setting has been reported (Wang et al., 2000; Tarn and Huang, 2002). Investigation via the state space formalism may prove to be effective in treating more general problems.

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Appendix A. Derivation of Eqs. (22)–(24)

When the stress and temperature fields are independent of x_3, Eqs. (7) and (9) may be expressed as

$$\partial_2\mathbf{u} = -[(\mathbf{K}_3 + \mathbf{C}_{22}^{-1}\mathbf{C}_{12}^T\mathbf{K}_1)\partial_1 + (\mathbf{K}_4 + \mathbf{C}_{22}^{-1}\mathbf{C}_{12}^T\mathbf{K}_2)\partial_3]\mathbf{u} + \mathbf{g}_1, \tag{A.1}$$

$$[\mathbf{K}_1^T\widetilde{\mathbf{C}}_{11}\mathbf{K}_1\partial_{11} + (\mathbf{K}_1^T\widetilde{\mathbf{C}}_{11}\mathbf{K}_2 + \mathbf{K}_2^T\widetilde{\mathbf{C}}_{11}\mathbf{K}_1)\partial_{13} + \mathbf{K}_2^T\widetilde{\mathbf{C}}_{11}\mathbf{K}_2\partial_{33}]\mathbf{u} = \mathbf{g}_2, \tag{A.2}$$

$$(\mathbf{K}_1\partial_1 + \mathbf{K}_2\partial_3)\mathbf{u} = \mathbf{g}_3, \tag{A.3}$$

where $\mathbf{g}_i$ are known functions of x_1 and x_2. Without loss in generality the expressions need not be written out.

Eq. (A.2) suggests that u_1, u_2, u_3 are at most power functions of x_3 of degree two. Thus

$$u_1 = f_1(x_1,x_2)x_3^2 + f_2(x_1,x_2)x_3 + f_3(x_1,x_2), \tag{A.4}$$

$$u_2 = f_4(x_1,x_2)x_3^2 + f_5(x_1,x_2)x_3 + f_6(x_1,x_2), \tag{A.5}$$

$$u_3 = f_7(x_1,x_2)x_3^2 + f_8(x_1,x_2)x_3 + f_9(x_1,x_2), \tag{A.6}$$

where f_i are yet unknown functions. Substituting them into Eq. (A.3) yields

$$f_1 = c_1 = \text{constant}, \qquad f_2 = f_2(x_2), \qquad f_7 = 0, \qquad \partial_1 f_8(x_1,x_2) = -2c_1. \tag{A.7}$$

Integrating Eq. $(\text{A.7})_4$ gives

$$f_8 = -2c_1x_1 + f(x_2). \tag{A.8}$$

Substituting Eqs. (A.7) and (A.8) in (A.1), equating the terms of x_3 on both sides produces

$$f_4(x_1,x_2) = c_4 = \text{constant}, \tag{A.9}$$

$$\partial_2 f_2(x_2) = -\partial_1 f_5(x_1,x_2), \qquad \partial_2 f_5(x_1,x_2) = 0, \qquad \partial_2 f(x_2) = -2c_4, \tag{A.10}$$

$$\partial_2 f_3(x_1,x_2) = h_1, \qquad \partial_2 f_6(x_1,x_2) = h_2, \qquad \partial_2 f_9(x_1,x_2) = h_3, \tag{A.11}$$

where h_1 are known functions of x_1 and x_2, from which

$$f_2 = -c_2 x_2 + \omega_2, \qquad f_5 = c_2 x_1 - \omega_1, \qquad f(x_2) = -2c_4 x_2 - c_5, \tag{A.12}$$

$$f_3 = u(x_1,x_2) - \omega_3 x_2 + u_0, \qquad f_6 = v(x_1,x_2) + \omega_3 x_1 + v_0, \tag{A.13}$$

$$f_9 = w(x_1,x_2) - \omega_2 x_1 + \omega_1 x_2 + w_0, \tag{A.14}$$

where the linear terms of x_1 and x_2 represent rigid body displacements.

Substituting f_i in Eqs. (A.4)–(A.6), slightly changing the notations, there follows the displacement field given by Eqs. (22)–(24).

Appendix B. An alternative formulation

The three-dimensional equations can be rearranged in matrix forms by grouping the stress into inplane and antiplane components. The state equation is formulated as

$$\frac{\partial}{\partial x_3}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\tau}_z\end{bmatrix} = \begin{bmatrix}\mathbf{L}_{11} & \mathbf{L}_{12}\\ \mathbf{L}_{21} & \mathbf{L}_{11}^T\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\tau}_z\end{bmatrix} + \begin{bmatrix}\mathbf{C}_{zz}^{-1}\boldsymbol{\beta}_z\\ \mathbf{D}_1^T\tilde{\boldsymbol{\beta}}_p\end{bmatrix}T - \begin{bmatrix}\mathbf{0}\\ \mathbf{F}\end{bmatrix}, \tag{B.1}$$

and the output equation as

$$\boldsymbol{\tau}_p = \begin{bmatrix}\widetilde{\mathbf{C}}_{pp}\mathbf{D}_1 & \mathbf{C}_{pz}\mathbf{C}_{zz}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\tau}_z\end{bmatrix} - \tilde{\boldsymbol{\beta}}_p T, \tag{B.2}$$

where

$$\mathbf{u} = [\,u_1 \quad u_2 \quad u_3\,]^T, \qquad \mathbf{F} = [\,F_1 \quad F_2 \quad F_3\,]^T,$$

$$\boldsymbol{\tau}_p = [\,\sigma_{11} \quad \sigma_{22} \quad \sigma_{12}\,]^T, \qquad \boldsymbol{\tau}_z = [\,\sigma_{13} \quad \sigma_{23} \quad \sigma_{33}\,]^T,$$

$$\boldsymbol{\beta}_p = [\,\beta_1 \quad \beta_2 \quad \beta_6\,]^T, \qquad \boldsymbol{\beta}_z = [\,\beta_5 \quad \beta_4 \quad \beta_3\,]^T,$$

$$\mathbf{L}_{11} = -\mathbf{D}_2 - \mathbf{C}_{zz}^{-1}\mathbf{C}_{pz}^T\mathbf{D}_1, \qquad \mathbf{L}_{12} = \mathbf{C}_{zz}^{-1}, \qquad \mathbf{L}_{21} = -\mathbf{D}_1^T\widetilde{\mathbf{C}}_{pp}\mathbf{D}_1,$$

$$\mathbf{D}_1 = \mathbf{K}_1\partial_1 + \mathbf{K}_2\partial_2, \qquad \mathbf{D}_2 = \mathbf{K}_3\partial_1 + \mathbf{K}_4\partial_2,$$

$$\widetilde{\mathbf{C}}_{pp} = \mathbf{C}_{pp} - \mathbf{C}_{pz}\mathbf{C}_{zz}^{-1}\mathbf{C}_{pz}^T, \qquad \tilde{\boldsymbol{\beta}}_p = \boldsymbol{\beta}_p - \mathbf{C}_{pz}\mathbf{C}_{zz}^{-1}\boldsymbol{\beta}_z,$$

$$\mathbf{C}_{pp} = \begin{bmatrix}c_{11} & c_{12} & c_{16}\\ c_{12} & c_{22} & c_{26}\\ c_{16} & c_{26} & c_{66}\end{bmatrix}, \qquad \mathbf{C}_{zz} = \begin{bmatrix}c_{55} & c_{45} & c_{35}\\ c_{45} & c_{44} & c_{34}\\ c_{35} & c_{34} & c_{33}\end{bmatrix}, \qquad \mathbf{C}_{pz} = \begin{bmatrix}c_{15} & c_{14} & c_{13}\\ c_{25} & c_{24} & c_{23}\\ c_{56} & c_{46} & c_{36}\end{bmatrix},$$

$$\mathbf{K}_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \qquad \mathbf{K}_2 = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}, \qquad \mathbf{K}_3 = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \qquad \mathbf{K}_4 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}.$$

For the generalized plane problem the state equation and the output equation become

$$\begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{11}^T \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{u}} \\ \boldsymbol{\tau}_z \end{bmatrix} = \begin{bmatrix} \mathbf{P} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{C}_{zz}^{-1}\boldsymbol{\beta}_z \\ \mathbf{D}_1^T\tilde{\boldsymbol{\beta}}_p \end{bmatrix} T + \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_p \end{bmatrix}, \tag{B.3}$$

$$\boldsymbol{\tau}_p = \begin{bmatrix} \widetilde{\mathbf{C}}_{pp}\mathbf{D}_1 & \mathbf{C}_{pz}\mathbf{C}_{zz}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{u}} \\ \boldsymbol{\tau}_z \end{bmatrix} - \tilde{\boldsymbol{\beta}}_p T, \tag{B.4}$$

where

$$\tilde{\mathbf{u}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \qquad \mathbf{f}_p = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix}, \qquad \mathbf{P} = \varepsilon \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \vartheta \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} + b_1 \begin{bmatrix} 0 \\ 0 \\ x_1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 0 \\ x_2 \end{bmatrix}.$$

The homogeneous solution to Eq. (B.3) takes the form

$$\tilde{\mathbf{u}} = \mathbf{U}F(z), \qquad \boldsymbol{\tau}_z = \mathbf{S}F'(z), \qquad z = x_1 + px_2. \tag{B.5}$$

Substituting Eq. (B.5) in Eq. (B.3) yields

$$\begin{bmatrix} \mathbf{A}_1 + p\mathbf{A}_2 & \mathbf{C}_{zz}^{-1} \\ \mathbf{B}_1 + p\mathbf{B}_2 + p^2\mathbf{B}_3 & \mathbf{A}_1^T + p\mathbf{A}_2^T \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{S} \end{bmatrix} = \mathbf{0}, \tag{B.6}$$

where

$$\mathbf{A}_1 = -\mathbf{C}_{zz}^{-1}(\mathbf{C}_{zz}\mathbf{K}_3 + \mathbf{C}_{pz}^T\mathbf{K}_1), \qquad \mathbf{A}_2 = -\mathbf{C}_{zz}^{-1}(\mathbf{C}_{zz}\mathbf{K}_4 + \mathbf{C}_{pz}^T\mathbf{K}_2),$$

$$\mathbf{B}_1 = -\mathbf{K}_1^T\widetilde{\mathbf{C}}_{pp}\mathbf{K}_1, \qquad \mathbf{B}_2 = -\mathbf{K}_1^T\widetilde{\mathbf{C}}_{pp}\mathbf{K}_2 - \mathbf{K}_2^T\widetilde{\mathbf{C}}_{pp}\mathbf{K}_1, \qquad \mathbf{B}_3 = -\mathbf{K}_2^T\widetilde{\mathbf{C}}_{pp}\mathbf{K}_2.$$

Expressing $\mathbf{S}$ in terms of $\mathbf{U}$ using $(\text{B.6})_1$ gives

$$\mathbf{S} = [\mathbf{C}_{zz}\mathbf{K}_3 + \mathbf{C}_{pz}^T\mathbf{K}_1 + p(\mathbf{C}_{zz}\mathbf{K}_4 + \mathbf{C}_{pz}^T\mathbf{K}_2)]\mathbf{U}. \tag{B.7}$$

Substituting it in $(\text{B.6})_2$ leads to

$$[\mathbf{G}_1 + p(\mathbf{G}_2 + \mathbf{G}_2^T) + p^2\mathbf{G}_3]\mathbf{U} = 0, \tag{B.8}$$

where

$$\begin{aligned} \mathbf{G}_1 &= \mathbf{K}_3^T\mathbf{C}_{pz}^T\mathbf{K}_1 + \mathbf{K}_1^T\mathbf{C}_{pz}\mathbf{K}_3 + \mathbf{K}_3^T\mathbf{C}_{zz}\mathbf{K}_3 + \mathbf{K}_1^T\mathbf{C}_{pp}\mathbf{K}_1, \\ \mathbf{G}_2 &= \mathbf{K}_1^T\mathbf{C}_{pp}\mathbf{K}_2 + \mathbf{K}_1^T\mathbf{C}_{pz}\mathbf{K}_4 + \mathbf{K}_3^T\mathbf{C}_{pz}^T\mathbf{K}_2 + \mathbf{K}_3^T\mathbf{C}_{zz}\mathbf{K}_4, \\ \mathbf{G}_3 &= \mathbf{K}_4^T\mathbf{C}_{pz}^T\mathbf{K}_2 + \mathbf{K}_2^T\mathbf{C}_{pz}\mathbf{K}_4 + \mathbf{K}_4^T\mathbf{C}_{zz}\mathbf{K}_4 + \mathbf{K}_2^T\mathbf{C}_{pp}\mathbf{K}_2. \end{aligned}$$

A little manipulation shows that $\mathbf{G}_1 = \mathbf{Q}$, $\mathbf{G}_2 = \mathbf{R}$, $\mathbf{G}_3 = \mathbf{T}$. Non-trivial solution to Eq. (B.8) exists if the determinant of the coefficient matrix vanishes. This results in exactly the sextic equation of Stroh for p, but does not bring forth the eigen relation.

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